

INTERPOLATING BETWEEN CHOICES FOR THE APPROXIMATE INTERMEDIATE VALUE THEOREM

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ABSTRACT. This paper proves the approximate intermediate value theorem, constructively and from notably weak hypotheses: from pointwise rather than uniform continuity, without assuming that reals are presented with rational approximants, and without using countable choice. The theorem is that if a pointwise continuous function has both a negative and a positive value, then it has values arbitrarily close to 0. The proof builds on the usual classical proof by bisection, which repeatedly selects the left or right half of an interval; the algorithm here selects an interval of half the size in a continuous way, interpolating between those two possibilities.

1. INTRODUCTION

On MathOverflow [Shu16], Mike Shulman asked: “Can the approximate Intermediate Value Theorem be proven constructively about an arbitrary (pointwise) continuous function f , without using any form of choice or excluded middle (e.g. in the mathematics valid in any elementary topos with a natural numbers object)?”

Here, the approximate Intermediate Value Theorem is

Theorem 1.1. *If f is pointwise continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then for every $\epsilon > 0$, there is some x with $|f(x)| < \epsilon$.*

The proof below answers “Yes” to Shulman’s question, satisfying four notable constraints. The first is that the proof should be constructive, in particular using only intuitionistic logic and not relying on the principle of excluded middle. This puts the standard intermediate value theorem, in which $f(x) = 0$, out of reach [BR, chapter 6.2].

The second constraint is that the proof should use pointwise rather than uniform continuity. Under uniform continuity, there is a δ corresponding to the ϵ , there is a sequence of x ’s going from a to b in steps of less than δ , and one of those x ’s will have $|f(x)| < \epsilon$.

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This is the proof in Bishop and Bridges [BB85, theorem 4.8, p. 40], but pointwise continuity does not constructively imply uniform continuity.

The third constraint is that the proof should not use countable choice. Under countable choice, it suffices in the proof below to let $d_n = 0$ if $f(c_n) < \epsilon$ or $d_n = 1$ if $f(c_n) > -\epsilon$, but without countable choice, such a sequence is ill-defined.

The fourth constraint is that the proof should not assume that reals are presented with rational approximants. If a real number is defined as a sequence of rationals, then d_n can be defined as 0 if the n^{th} rational approximation to $f(c_n)$ is negative, or 1 otherwise. However, this operation does not respect the relation of equality on reals, and it is not appropriate for reals defined as equivalence classes of sequences or as Dedekind cuts.

Since the proof below satisfies these constraints, it is of interest for topoi in which countable choice fails and with reals defined by Dedekind cuts (*e.g.* [BR87, chapter 7.3]). It also strengthens some of Schuster's results [Sch03]. It reduces to the usual classical proof by bisection if the definition of d_n is replaced by $d_n = 0$ if $f(c_n) < 0$ and $d_n = 1$ otherwise, choosing either the left half or the right half of the interval at each step. The key difference from that proof is that the algorithm here selects an interval of half the size in a continuous way, interpolating between those two possibilities.

2. PROOF OF THE THEOREM

Define the following inductively:

$$\begin{aligned} a_1 &= a \\ b_1 &= b \\ c_n &= (a_n + b_n)/2 \\ d_n &= \max(0, \min(\frac{1}{2} + \frac{f(c_n)}{\epsilon}, 1)) \\ a_{n+1} &= c_n - d_n(b - a)/2^n \\ b_{n+1} &= b_n - d_n(b - a)/2^n \end{aligned}$$

Then $b_n - a_n = (b - a)/2^{n-1}$. So the c_n 's converge to some c .

Claim 2.1. For any $m \in \mathbb{N}$, either (i) $\exists j \leq m, |f(c_j)| < \epsilon$ or (ii) $f(a_m) < 0$ and $f(b_m) > 0$.

Proof of Theorem from Claim. By pointwise continuity at c , let δ be such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. Choose m such that $|c - c_m| < \delta/2$ and $(a - b)/2^m < \delta/2$, and apply the claim.

In the first case of the claim, the theorem is immediate.

In the second case of the claim,

$$|c - a_m| \leq |c - c_m| + |c_m - a_m| < \delta, \text{ so } |f(c) - f(a_m)| < \epsilon$$

$$|c - b_m| \leq |c - c_m| + |c_m - b_m| < \delta, \text{ so } |f(c) - f(b_m)| < \epsilon$$

So $f(c)$ is within ϵ of both a negative and a positive number, and $|f(c)| < \epsilon$.

Proof of Claim, by induction on m . The base case is given by $f(a) < 0, f(b) > 0$.

Now assume the claim for m . In case (i), for some $j < m$, $|f(c_j)| < \epsilon$, and the inductive step is trivial.

In case (ii), use trichotomy with either $f(c_m) < -\epsilon/2$, $|f(c_m)| < \epsilon$, or $f(c_m) > \epsilon/2$. If $|f(c_m)| < \epsilon$, then the inductive step is again trivial. If $f(c_m) > \epsilon/2$, then

$$d_m = 1$$

$$\begin{aligned} a_{m+1} &= a_m, \text{ so } f(a_{m+1}) < 0 \\ b_{m+1} &= c_m, \text{ so } f(b_{m+1}) > 0 \end{aligned}$$

If $f(c_m) < -\epsilon/2$, then

$$\begin{aligned} d_m &= 0 \\ a_{m+1} &= c_m, \text{ so } f(a_{m+1}) < 0 \\ b_{m+1} &= b_m, \text{ so } f(b_{m+1}) > 0 \end{aligned}$$

This establishes the claim and the theorem. □

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